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SINE – COSINE METHOD FOR SOLVING SOME NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Non-linear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Non-linear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of non-linear evolution equations have been proposed. A variety of powerful methods, such as, sine-cosine method, sine-cosine method, hyperbolic function method, Jacobi elliptic function expansion method, F-expansion method, and the First Integral method. The sine-cosine method has been used to solve different types of nonlinear systems of PDEs. The Sine-Cosine function algorithm is applied for solving nonlinear partial differential equations. The method was used to obtain the exact solutions for different types of nonlinear partial differential equations such as, The $K(n+1, n+1)$ equation, Schrödinger-Hirota equation, Gardner equation, the modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation which are the important Soliton equations. Based on three methods, exact solutions are obtained for cubic Boussinesq and modified Boussinesq equations. These are sine method, cosine method, and sine-cosine method. The obtained solutions contain solitary waves. The results reveal are very effective, convenient and quite accurate to such types of partial differential equations comparing with other methods. Sine - cosine method to obtain a new traveling wave solutions of a generalized Kawahara equation and we obtain a new exact solution for Hunter - Saxton equation by direct integration. The traveling wave solutions for the above equation are presented to obtain novel exact solutions for the same equations. These solutions plays an important role in the modeling of many physical phenomena such as plasma waves, magento – acoustic wave.

Key words: Non-linear evolution equations, Sine – Cosine method, Schrödinger-Hirota equation, Gardner equation, Burger's-Fisher equation, and Cubic modified Boussinesq equation.

1. Introduction

Non-linear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Non-linear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of non-linear evolution equations have been proposed. A variety of powerful methods, such as, Tanh-sech method, Extended tanh method, Hyperbolic function method and Jacobi elliptic function expansion method. F-expansion method, and the First Integral method. The sine-cosine method, has been used to solve different types of nonlinear systems of PDEs.

The Sine-Cosine function algorithm is applied for solving nonlinear partial differential

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equations. The method is used to obtain the exact solutions for different types of nonlinear partial differential equations such as. The $K\ (n+\ 1.n+1)$ equation. Schrodinger-Hirota equation, Gardner equation, the modified KdV equation, perturbed Burgers equation, general Burger's – Fisher equation and Cubic modified Boussinesq equation which are the important Soliton equations.

Based on three mothods, exact solution are obtained for cubic Boussinesq and modified Boussinesq equations. These are tanhmethoc, sech method, and sine-cosine method. The obtained solutions contain solitary waves. The results reveal are very effective, convenient and quite accurate to such types of partial differential equations comparing with other methods. Sine-cosine method to obtain a new traveling wave solutions of a generalized Kawahara equation and we obtain a new exact solution for Hunter-Saxton equation by direct integration. The traveling wave solutions for the above equation are presented to obtain novel exact solutions for the same equations. These solutions play an important role in the modeling of many physical phenomena such as plasms waves, magento-acoustic wave.

Problem Statement

In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of non-linear evolution equations have been proposed. A variety of powerful methods.

Study Objective

The main object of this research solution of some none linear partial differential equations on two methods, exact solution are obtained for some non-liner p equations. These are sine method, cosine method obtained solutions contain solitary waves. The results reveal are very effective, convenient and quite accurate to such types of partial differential equations comparing with other methods. Sine-cosine method to obtain a new traveling wave**.**

Lay out

Section 1, Present introduction problem Statement and objective. In section 2, Sine-Cosine method is produced and devoted to employed the method on the $K(n+1,n+1)$ equation, Schrödinger-Hirota equation, Gardner equation, modified KdV equation. In section 3, Find the new solution of Cubic modified Boussinesq equation. Generalized Kawahara equation and Hunter - Saxton equation by direct integration and also provide. The brief conclusion of this section 4 Conclusion and Recommendation. Finally, some references are listed in the end.

Literature review

Non-linear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Non-linear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in non-linear wave equations (Hirota, 2004). In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, sine-cosine method Wazwaz (2004) extended sine method Wazwaz (2006), hyperbolic function method Zhang (2001), Jacobi elliptic function expansion method Zhang (2005), F-expansion method Ali (2007), and the First Integral method Hirota (2007) . The sine-cosine method Wazwaz (2004) has been used to solve different types of nonlinear systems of PDEs.

The first part explains the proposed method, while the second part contains the applications. The aim of this paper was to find new exact solutions of the $K(n+1,n+1)$ equation, Schrödinger-Hirota equation, Gardner equation, modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation by the sine-cosine method.

The Sine-Cosine Function Method

Consider the nonlinear partial differential equation in the form

$$F(u,u_t,u_{tt},u_{xx},u_{yy},\dots\dots\dots)=0 \quad (1$$

Where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation Eq. (1). We use the transformations,

$$u(x, y, t) = f(\xi) \qquad (2)$$

Where $\xi = x + y - \lambda t$ This enables us to use the following changes:

$$\frac{\partial}{\partial t} = -\lambda \frac{d}{d\xi}(\cdot), \frac{d}{dx}(\cdot) = \frac{d}{d\xi}(\cdot), \frac{\partial}{d\xi}(\cdot)$$

Using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots\dots\dots) = 0$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form: 16, 17

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu} \qquad (3)$$

Or in the form (5)

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu}$$

Where α, μ, and β are parameters to be determined, and c are the wave number and the wave speed, respectively We use Wazwaz (2004)

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$$

$$f'(\xi) = \alpha\beta\mu \sin^{\beta-1}(\mu\xi)\cos(\mu\xi) \qquad (6)$$

$$F''(\xi) = -\alpha\beta M \sin^{\beta-1}(M\xi)\sin(M\xi) + \alpha\beta M^2(\beta-1)\sin^{\beta-2}(M\xi)cos^2(M\xi)$$

$$= -\alpha\beta M^2 \sin^{\beta}(M\xi) + \alpha\beta M^2(\beta-1)\sin^{\beta-2}(M\xi)[1\sin^2(M\xi)]$$

$$-\alpha\beta M^2 \sin^{\beta}(M\xi) + \alpha\beta M^2(\beta-1)\sin^{\beta-2}(M\xi) - \alpha\beta M^2(\beta-1)\sin^{\beta}(M\xi)$$

$$= -\alpha\beta M^2 \sin^{\beta}(M\xi) + \alpha\beta M^2(\beta-1)\sin^{\beta-2}(M\xi) - \alpha\beta^2 M^2 \sin^{\beta}(M\xi) + \alpha\beta M^2 \sin^{\beta}(M\xi)$$

$$= \alpha\beta M^2(\beta-1)\sin^{\beta-2}(M\xi) - \alpha\beta^2 M^2 \sin^{\beta}(M\xi)$$

$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \sin^{\beta}(\mu\xi)$$

And their derivative. Or use

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$$

$$f'(\xi) = -\alpha\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi) \qquad (7)$$

$$-\alpha\beta M^2 \cos^{\beta-1}(M\xi)\cos(M\xi) + \alpha\beta M^2(\beta-1)\cos^{\beta-2}(M\xi)\sin^2(M\xi)$$

$$= -\alpha\beta M^2 \cos^{\beta}(M\xi) + \alpha\beta M^2(\beta-1)\cos^{\beta-2} - \alpha\beta M^2(\beta-1)\cos^{\beta}(M\xi)$$

$$-\alpha\beta M^2 \cos^{\beta}(M\xi) + \alpha\beta M^2(\beta-1)\cos^{\beta-2}(M\xi)$$

$$-\alpha\beta^2 M^2 \cos^{\beta}(M\xi) + \alpha\beta M^2 \cos^{\beta}(M\xi)$$

$$= \alpha\beta M^2(\beta-1)\cos^{\beta-2}(M\xi) - \alpha\beta^2 M^2 \cos^{\beta}(M\xi)$$

$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \cos^{\beta}(\mu\xi)$$

$$f'''(\xi) = -\alpha\beta(\beta-1)(\beta-2)\mu^3 \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) + \alpha\beta^3\mu^3 \cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$

and so on. We substitute Eq.(6) or Eq.(7) into the reduced equation Eq.(4), balance the terms of the sine functions when Eq.(6) are used, or balance the terms of the cosine functions when Eq.(7) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^k(\mu\xi)$ or $\cos^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's α, μ and β, and solve the subsequent system.

2. Applications

The first part explains the proposed method, while the second part contains the applications. The aim of this paper is to find new exact solutions of the $K(n+1, n+1)$ equation, Schrödinger-Hirota equation, Gardner equation, modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation by the Sine-cosine method.

2.1 Gardner equation

Let us consider the Gardner equation Ali (2007)

$$u_t - 6(u + \varepsilon^2 u^2)u_x + u_{xxx} = 0 \qquad (8)$$

This equation known as the mixed KdV-mKdV equation is very widely studied in various areas of Physics that includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others.

We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. Equation (8) transforms to the ODE:

$$-k\lambda u' - 3k(u^2)' - 2\varepsilon^2 k(u^3)' + k^3 u''' = 0$$

Integrating Eq.(8) once with zero constant to get the following ordinary differential equation:

$$\lambda u + 3u^2 + 2\varepsilon^2 u^3 - k^2 u'' = 0 \qquad (10)$$

Seeking the solution in Eq.(7)

$$\lambda\alpha\cos^{\beta}(\mu\xi) + 3\alpha^2\cos^{2\beta}(\mu\xi) + 2\varepsilon^2\alpha^3\cos^{3\beta}(\mu\xi) - \alpha\beta(\beta-1)k^2\mu^2\cos^{\beta-2}(\mu\xi) + \alpha\beta^2\mu^2\cos^{\beta}(\mu\xi) = 0 \qquad (11)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta(\beta-1)(\beta-2) \neq 0$$

$$3\beta = \beta - 2 \rightarrow \beta = -1 \qquad (12)$$

Substituting Eq. (11) into Eq. (12) to get:

$$\lambda\alpha\cos^{-1}(\mu\xi) + 3\alpha^2\cos^{-2}(\mu\xi) + 2\varepsilon^2\alpha^3\cos^{-3}(\mu\xi) - 2\alpha k^2\mu^2\cos^{-3}(\mu\xi) + \alpha\mu^2 k^2\cos^{--1}(\mu\xi) = 0 \qquad (13)$$

Equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

$$\cos^{-3}(\mu\xi): 2\varepsilon^2\alpha^3 - 2\alpha k^2\mu^2 = 0$$

$$\cos^{-2}(\mu\xi): 3\alpha^2 = 0$$

$$\cos^{-1}(\mu\xi): \lambda\alpha + \alpha\mu^2 k^2 = 0 \qquad (14)$$

By solving the algebraic system (14), we get,

$$\beta = -1, \quad \lambda = -\mu^2 k^2, \quad \alpha = \mp\frac{k\,\mu}{\varepsilon} \qquad (15)$$

Then by substituting Eq. (14) into Eq. (7), the exact soliton solution of Eq.(29)be in the form

$$u(x,t) = \mp\frac{k\,\mu}{\varepsilon}\sec\!\big(\mu\, k(x+\mu^2k^2t)\big), 0 < \mu\, k(x+\mu^2k^2t) < \pi \qquad (16)$$

For, $\mu = k = \varepsilon = 1$, then Eq.(16) becomes:

$$u(x,t) = \sec(x+t) \quad , \quad 0 < (x+t) < \pi \qquad (17)$$

2.2 Dispersive equation

Consider the (1+1)-dimensional nonlinear dispersive equation Wazwaz (2004)

$$u_t - \delta u^2 u_x + u_{xxx} = 0 \qquad (18)$$

Where δ is a nonzero positive constant. This equation is called the modified KdV equation Elsayed *et al*. (2004), which arises in the process of understanding the role of non-linear dispersion and in the formation of structures like liquid drops, and it exhibits compaction solutions with compact support. To find the traveling wave solutions of Eq. (18), He *et al*. (2004) used the Exp-function method, and Wazwaz (2004) used ⁄ Expansion method.

Let us now solve Eq. (18) by the proposed method. We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. Equation (18) transforms to the ODE:

$$-k\lambda u' - \frac{\delta}{3}k(u^3)' + k^3 u''' = 0 \qquad (1$$

Integrating Eq.(19) once with zero constant to get the following ordinary differential equation:

$$\lambda u + \frac{\delta}{3}u^3 - k^2 u'' = 0 \qquad (20)$$

Seeking the solution in Eq.(7)

$$\lambda\alpha\cos^{\beta}(\mu\xi) + \frac{\delta}{3}\alpha^3\cos^{3\beta}(\mu\xi) - \alpha\beta(\beta-1)k^2\mu^2\cos^{\beta}(\mu\xi) + \alpha\beta^2\mu^2k^2\cos^{\beta}(\mu\xi) = 0 \qquad (21)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$3\beta = \beta - 2 \to \beta = -1$$

$$\cos^{-3}(\mu\xi) : \frac{\delta}{3}\alpha^3 - 2\alpha k^2\mu^2 = 0$$

$$\cos^{-1}(\mu\xi) : \lambda\alpha + \alpha\mu^2k^2 = 0 \qquad (22)$$

By solving the algebraic system (22), we get,

$$\beta = -1, \quad \lambda = -\mu^2k^2, \quad \alpha = \mp\sqrt{\frac{6}{\delta}}\,k\mu \qquad (23)$$

Then, by substituting Eq. (23) into Eq. (7) , the exact soliton solution of Eq.(37) can be written in the form

$$u(x,t) = \mp\sqrt{\frac{6}{\delta}}\,k\mu\ \sec\big(\mu k(x+\mu^2k^2t)\big), \qquad 0 < \mu k(x+\mu^2k^2t) < \pi \qquad (24)$$

2.3 The $k(n+1, n+1)$ equation

Let us consider the following $k(n+1, n+1)$ equation [19]:

$$u_t + a(u^{n+1})_x + b(u(u^n)_{xx})_x = 0 \qquad (25)$$

Where a and b are nonzero constants. We introduce the transformation $\xi = k(x-\lambda t)$, where k, and λ are real constants. The traveling wave variable ξ permits us converting Eq. (25) into the following ODE:

$$u(x,t) = k[x-\lambda t]$$

$$u_t = \frac{\partial u}{\partial \delta}.\frac{\partial \delta}{\partial t}$$

$$\frac{\partial u}{\partial \delta} = u'\,\frac{\partial \delta}{\partial t} = -\lambda k$$

$$u_t = -\lambda k u' = \frac{-\lambda k u'}{ku^{n-2}}$$

$$(u^{n+1})_x = a(n+1).u^n.u'.k = \frac{a(n+1).u^n u' k}{ku^{n-2}}$$

$$(u_x^n) = \frac{kn(u)^{n-1}\lambda}{ku^{n-2}}$$

$$u_{xx} = k^2n(n-1)(u)^{n-1}u'' + k^2n(n-1)u^{n-2}(u')^2$$

$$(u_{xx}) = k^3n(n-1)(u)^{n-2}u'uu'' + k^3n(n-1)u^{n-1}u'u'' + k^3n(u)^{n-1}uu''' + k^3n(n-1)(n-2)(u^{n-3})u(u)^3$$

$$= b[k^2n(n-1)uu'u'' + k^2nuu'u'' + k^2nu^2u''' + k^2n(n-1)(n-2)u^{-1}uu'' + knu^2u'' + k^2n(n-1)(n-1)u^{n-1}u(u')^3 + k^2n(n-1)(u')^3 + 2k^2n(n-1)uu'u'']$$

$$= -\lambda u'u^{2-n} + a(n+1)u^2u' + bnk^2u^2u''' + bk^2n[3n-2]uu'u'' + bn(n-1)^2k^2u'^3 = 0$$

$$\lambda\beta M\alpha^{3-n} - bnk^2\alpha^3B(B-1)(B-2)M^3 - bk^2n(3n-2)\alpha^3\beta^2(\beta-1)M^3 - bn(n-1)^2k^2\beta^3M^3\alpha^3 = 0$$

$$-\lambda\, u\, u^{2-n} + a(n+1)\, u^2u' + bn\, k^2u^2u''' + b\, k^2n[3n-2]u\, u'u'' + bn(n-1)^2k^2u'^3 = 0 \qquad (26)$$

Seeking the solution in Eq. (7)

$$\lambda\beta\mu\,\alpha^{3-n}\cos^{(3-n)\beta-1}(\mu\xi)\sin(\mu\xi)-a(n+1)\alpha^3\beta\mu\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$-bnk^2\alpha^3\beta(\beta-1)(\beta-2)\mu^3\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)+bnk^2\alpha^3\beta^3\mu^3\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$-bk^2n(3n-2)\alpha^3\beta^2(\beta-1)\mu^3\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)+bk^2n(3n-2)\alpha^3\beta^3\mu^3\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$-bn(n-1)^2k^2\beta^3\mu^3\alpha^3\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)$$
$$=0 \qquad (27)$$

From Eq.(10), equating exponents $(3-n)\beta-1$ and $3\beta-3$ yield

$$(3-n)\beta-1=3\beta-3 \qquad (28)$$

So that

$$\beta=\frac{2}{n} \qquad (29)$$

Thus setting coefficients of Eq.(27) to zero yields the following system of equations:

$$\lambda\beta\mu\alpha^{3-n}-bnk^2\alpha^3\beta(\beta-1)(\beta-2)\mu^3-bk^2n(3n-2)\alpha^3\beta^2(\beta-1)\mu^3-bn(n-1)^2k^2\beta^3\mu^3\alpha^3=0$$

$$-a(n+1)\alpha^3\beta\mu+bnk^2\alpha^3\beta^3\mu^3+bk^2n(3n-2)\alpha^3\beta^3\mu^3=0 \qquad (30)$$

By solving the algebraic system (30), we get,

$$\lambda\alpha^{-n}=bnk^2(\beta-1)(\beta-2)M^2-bk^2n(3n-2)\beta(\beta-1)M^2-bn[n-1]^2k^2\beta^2M^2=0$$

$$bnk^2M^2[(\beta-1)(\beta-2)-[3n-2]\beta(\beta-1)]-[n-1]^2\beta^2$$

$$M^2=\frac{a[n+1]}{\frac{4}{n}k^2b[3n-1]}$$

$$=\frac{n^2a(n+1)\left[\frac{2}{n}-1\right]\left[\frac{2}{n}-2\right]-[3n-2]\frac{2}{n}\left[\frac{2}{n}-1\right]-[n-1]^2\frac{4}{n^2}}{4[3n-1]}$$

$$=\frac{a[n+1][2-n][2-2n]-[3n-2][n-2]-[n-1]^24}{4[3n-1]}$$

$$=\frac{a[n+1][2n^2-7n-4]}{2(3n-1)}$$

$$=\frac{a[n+1][2n+1][n-4]}{2(3n-1)}$$

$$\alpha=\left\{\frac{2(3n-1)}{a(n+1)(2n+1)(n-4)}\lambda\right\}^{\frac{1}{n}},$$

$$-a[n+1]\alpha^3\beta M+bnk^2\alpha^3\beta^2M^3+$$
(1) $bk^2n(3n-2)\alpha^3\beta^3M^3=0$

$$-\alpha[n+1] + bk^2\beta^2M^2 + bk^2n[3n-2]\beta^2M^2 = 0$$

$$M^2[bnk^2\beta^2 + bk^2n[3n-2]\beta^2 = \alpha[n+1]$$

$$M^2 = \frac{\alpha[n+1]}{bk^2n\beta^2[1+3n-2]}$$

$$M^2 = \frac{\alpha[n+1]}{4\frac{k^2}{n}b[3n-1]}$$

$$M = \sqrt{\frac{\frac{a\,n(n+1)}{b\,(3n-1)}}{2k}} \qquad (31)$$

Then by substituting Eq. (31) into Eq. (7) , the exact soliton solution of Eq.(31) can be written in the form

$$u(x,t) = \left[\frac{2(3n-1)}{\alpha(n+1)(2n+1)(n-4)}\lambda\cos^2\left(\sqrt{\frac{a\,n(n+1)}{4b\,(3n-1)}}(x-\lambda t)\right)\right]^{\frac{1}{n}} \qquad (32)$$

2.4 The Schrödinger-Hirota equation

Consider the nonlinear The Schrödinger-Hirota equation which governs the propagation of optical Soliton in a dispersive optical fiber:

$$iq_t + \frac{1}{2}q_{xx} + |q|^2\,q + i\,\lambda q_{xxx} = 0 \qquad (33)$$

$$q_t = ex^{i\theta}u'(\xi).(-2\alpha k_0) + ie^{i\theta}.w$$

$$q_t = ie^{i\theta}u(\xi).\alpha + e^{i\theta}u'(\xi).k_0$$

$$q_{tt} = -e^{i\theta}u(\xi)\alpha^2 + ie^{i\theta}u'(\xi).k_0\alpha + ie^{i\theta}u'(\xi)\alpha k_0 + e^{i\theta}u''(\xi)k_0^2$$

$$= -\alpha^2e^{i\theta}u(\xi) + 2i\alpha k_0e^{i\theta}u'(\xi) + k_0^2e^{i\theta}u''(\xi)$$

$$q_{xxx} = i\alpha^3e^{i\theta}u(\xi) - \alpha^2k_0e^{i\theta}u'(\xi) - \alpha^2k_0e^{i\theta}u'(\xi) + 2i\alpha k_0^2e^{i\theta}u''(\xi) + i\alpha k^2e^{i\theta}u''(\xi) + k_0^3e^{i\theta}u'''(\xi)$$

$$= -i\alpha^3e^{i\theta}u(\xi) - 3\alpha^2k_0e^{i\theta}u'(\xi) + 3i\alpha k_0^2e^{i\theta}u''(\xi) + k_0^3e^{i\theta}u'''(\xi)$$

$$q = e^{i\theta}u.(\xi)q^2 = e^{2i\theta}u(\xi)^2$$

$$q.q^2 = e^{3i\theta}u(\xi)^3$$

This equation studied by Biswas *et al*. (2016) by the Ansatz method for bright and dark 1-soliton solution. The power law non-linearity was assumed. The equation was solved also by using the Tanh method. Introduce the transformations:

$$q(x,t) = e^{i\theta} \cdot u(\xi)\ , \theta = \alpha x + \omega t + \epsilon_0, \quad \xi = k_0(x - 2\alpha t + \mathcal{X}) \qquad (34)$$

Where $\alpha, \omega, \epsilon_0, k_0$, and $\mathcal{X}$ are real constants. Substituting Eq.(34) into Eq.(33) we obtain that $\alpha = -\frac{1}{3\lambda}$ and $u(\xi)$ satisfy into the ODE:

$$-\left(\frac{5}{54\lambda^2} + \omega\right)u(\xi) + \frac{3}{2}k_0 u''(\xi) + \left(u(\xi)\right)^3 = 0 \qquad (35)$$

Then we can write the following equation:

$$u'' + k_1 u^3 - k_2 u = 0 \qquad (36)$$

Where

$$k_1 = \frac{1}{\frac{3}{2}k_0^2}, \qquad k_2 = \frac{\left(\frac{5}{54\lambda^2} + \omega\right)}{\frac{3}{2}k_0^2} \qquad (37)$$

Seeking solutions of the form Eq.(6) we get:

$$\alpha\beta(\beta-1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \sin^{\beta}(\mu\xi) + k_1\alpha^3 \sin^{3\beta}(\mu\xi) - k_2\alpha \sin^{\beta}(\mu\xi) = 0 \qquad (38)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta - 2 = 3\beta$$

$$\alpha\beta(\beta-1)\mu^2 + k_1\alpha^3 = 0$$

$$-\alpha\beta^2\mu^2 - k_2\alpha = 0 \qquad (39)$$

By solving the algebraic system (24), we get,

$$\beta = -1,$$

$$-\alpha\beta^2 M^2 - k_2\alpha = 0$$

$$-\beta^2 M^2 = k_2$$

$$-M^2 = k_2$$

$$\mu = \pm i\sqrt{k_2},$$

$$\alpha\beta(\beta-1)M^2 + k_1\alpha^3 = 0$$

$$\beta(\beta-1)M^2 + k_1\alpha^2 = 0$$

$$\alpha^2 = \frac{-\beta(\beta-1)M^2}{k_1} = \frac{+1(-2)(k_2)}{k_1}$$

$$\alpha^2 = 2\frac{k_2}{k_1} \Rightarrow \alpha \pm \sqrt{\frac{2k_2}{k_1}}$$

$$\alpha = \pm\sqrt{\frac{2k_2}{k_1}} \qquad (40)$$

Then by substituting Eq. (40) into Eq. (6) , the exact soliton solution of equation Eq.(36) can be written in the form:

$$u(\xi) == \pm\sqrt{\frac{5}{27\lambda^2} + 2\beta}\,\csc\left(\pm i\sqrt{k_2}\,\xi\right) \qquad (41)$$

Or

$$u(\xi) = \mp\sqrt{\frac{5}{27\lambda^2} + 2\beta}\,\operatorname{csch}\left(\sqrt{k_2}\,\xi\right) \qquad (42)$$

Therefore

$$u(x,y,t) = \pm\sqrt{\left(\frac{5}{27\lambda^2} + 2\omega\right)}\operatorname{csch}\left(\frac{\sqrt{\left(\frac{5}{54\lambda^2} + \omega\right)}}{\frac{3}{2}k_0^2}k_0\left(x + \frac{2}{3\lambda}t + x\right)\right)e^{i\left(-\frac{1}{3\lambda}x + \omega t + \epsilon_0\right)} \qquad (43)$$

Solutions for the Boussinesq Equations

Section 1: Introduction

Non-linear evolution equations (NLEEs) have come a long way through. These NLEEs appear in various areas of Physics, Engineering, Biological Sciences, Geological Sciences and many other places. These equations arise of necessity. Subsequently they are studied in these various scientific contexts. There are various aspects of these NLEEs that are studied by various scientists and engineers as the need arises. Some of the commonly studied aspects are integrability, conservation laws, stochasticity, numerical solutions and many other aspects.

The non-linear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped cosine solutions and the kink-shaped sine solutions. The availability of these exact solutions, for those non-linear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. The investigation of exact solutions of NLPDEs plays an important role in the study of these phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions, such as Hirota's direct method (Hirota, 2004).

Sine-cosine method (Malfliet, 1993; Wazwaz 2006) extended sine method Ma and Fuchssteiner (1996), Fan (2000), Wazwaz (2005) and El-Wakil *et al*. (2007), Hyperbolic function method (Xia and Zhang 2001), Sine-cosine method (Wazwaz, 2004),Yusufoglu and Bekir (2006), F-expansion method (Zhang, 2006), and the transformed Rational Function Method (Ma and Lee, 2009).

The NLEEs equations that are going to be studied in this paper are the cubic Boussinesq and modified Boussinesq equations. The various analytical techniques that are going to be applied to these equations are tanh, sech, and sine-cosine methods. Subsequently, these solutions are supported by numerical simulations. Comparison of the present solutions is made with HPM and Padé technique. Excellent agreement is noted.

The Sine-Cosine Function Method

Consider the nonlinear partial differential equation in the form: (see Mitchell, 1980; Parkes,1998; Khater, 2002)

$$F\left(u, u_t, u_x, u_y, u_{tt}, u_{xy}, u_{yy}, \dots\dots\dots\right) = 0 \qquad (1)$$

Where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation Eq. (1) . We use the transformations,

$$\xi = x + y - \lambda t \qquad (2)$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(\cdot) = -\lambda \frac{d}{d\xi}(\cdot), \frac{\partial}{\partial x}(\cdot), \frac{\partial}{d\xi}(\cdot), \frac{\partial}{\partial y}(\cdot) = \frac{d}{d\xi}(\cdot) \qquad (3)$$

Using (2) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots\dots\dots.) = 0 \qquad (4)$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form: (Ali *et al*., 2007; Wazwaz, 2004).

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi), \qquad or \qquad f(\xi) = \alpha \cos^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu} \qquad (5)$$

Where $\alpha, \mu,$ and β are parameters to be determined, μ and λ are the wave number and the wave speed, respectively. Parks (1998) use

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$$
$$f'(\xi) = \alpha\beta\mu \sin^{\beta-1}(\mu\xi)\cos(\mu\xi) \qquad (6)$$
$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \sin^{\beta}(\mu\xi)$$
$$f'''^{(\xi)} = \alpha\beta(\beta-1)(\beta-2)\mu^3 \sin^{\beta-3}(\mu\xi)\cos(\mu\xi) - \alpha\beta^3\mu^3 \sin^{\beta-1}(\mu\xi)\cos(\mu\xi)$$

And their derivative. Or use

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$$
$$f'(\xi) = -\alpha\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi) \qquad (7)$$
$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \cos^{\beta}(\mu\xi)$$
$$f'''(\xi) = -\alpha\beta(\beta-1)(\beta-2)\mu^3 \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) + \alpha\beta^3\mu^3 \cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$

Substitute (6) or (7) into the reduced equation (4), balance the terms of the sine functions when (6) are used, or balance the terms of the cosine functions when (7) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^k(\mu k)$ or $\cos^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's α, μ and $\beta,$ and solve the subsequent system.

3. Applications

The tanh, sech , and sine-cosine methods are generalized on equations.

Example 1: Consider the cubic modified Boussinesq equation,

$$u_{tt} + u_{xxt} + \frac{2}{9}u_{xxxx} - (u^3)_{xx} = 0 \qquad (8)$$

$$u(x,t) = \cos^{\beta}\mu\xi \qquad (9)$$

The traveling wave hypothesis as given by

$$\xi = kx - \lambda t \qquad (10)$$

The nonlinear partial differential equation (8) is carried to an ordinary differential equation

$$\lambda^2 U'' - k^2\lambda U''' + \frac{2}{9}k^4 U'''' - 3k^2(U^2U')' = 0 \qquad (11)$$

Integrating Eq. (11) twice with zero constant and we postulate the tanh series, Eq (11) reduces to

$$\lambda^2 U - k^2\lambda U' + \frac{2}{9}k^4 U'' - k^2U^3 = 0 \qquad (12)$$

Sine-Cosine Method

By applying Sine-cosine method to solve Eq. (12), and seeking the solution in (7) then

$$\lambda^2\alpha\cos^{\beta}(\mu\xi) + k^2\lambda\ \ \alpha\beta\mu\cos^{\beta-1}(\mu\xi)\sin(\mu\xi) + \frac{2}{9}k^4\alpha\beta(\beta-1)\mu^2\cos^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2\cos^{\beta}(\mu\xi)$$
$$- k^2\alpha^3\cos^{3\beta}(\mu\xi) = 0 \qquad (13)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta - 2 = 3\beta, \quad \text{then} \quad \beta = -1$$
$$\lambda^2\alpha - \alpha\beta^2\mu^2 = 0$$
$$\frac{2}{9}k^4\alpha\beta(\beta-1)\mu^2 - k^2\alpha^3 = 0 \qquad (14)$$

By solving the algebraic system (14), we get,

$$\lambda = \mp\mu, \qquad \alpha = \frac{2}{3} \qquad (15)$$

Then by substituting (15) into Eq. (7) then, the exact soliton solution of equation (8) can be written in the form:

$$u_2(x,t) = \frac{2}{3}k\mu\sec\big(\mu(kx \mp \mu t)\big) \qquad (16)$$

For $k = \frac{3}{2}$, $\mu = 1$, then:

$$u_2(x,t) = \sec\left(\frac{3}{2}x \mp t\right) \qquad (17)$$

Example 2. Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation,

$$u_{tt} - u_{xxxx} - (u^3)_{xx} = 0 \qquad (18)$$

$$\xi = kx - \eta t\ u_t = -\eta u'$$

$$u_{tt} = \eta^2 u''(i)$$

$$u_x = ku' u_{xxxx} = k^4 u'''' \qquad (ii)$$

$$(u^3)_x = 3ku^2u'(u^3)_{xx} = 3k^2(u^2u')' \qquad (iii)$$

Mohamed *et al*. (2009) tried to solve Eq.(8) by applied the Homotopy Perturbation method and Padé approximants. The exact solution of Eq.(8) is;

$$u(x,t) = \sqrt{2}\,\mathrm{sech}(x-t) \qquad (19)$$

The nonlinear partial differential equation (18) is carried to an ordinary differential equation using the transformation

$$\xi = kx - \lambda t \qquad (20)$$

Then

$$\lambda^2 U'' - k^4 U'''' - 3k^2 (U^2 U')' = 0 \qquad (21)$$

Integrating Eq.(11) twice and assuming the constant of integration equal to zero, then

$$\lambda^2 U - k^4 U'' - k^2 U^3 = 0 \qquad (22)$$

Sine-Cosine Method

By applying Sine-cosine method to solve Eq.(12), and seeking the solution in (7) then

$$\begin{aligned} &-\lambda^2 \alpha\beta\mu \cos^{\beta}(\mu\xi)\sin(\mu\xi) \\ &\quad + k^4\left[\alpha\beta(\beta-1)(\beta-2)\mu^3 \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) - \alpha\beta^3\mu^3 \cos^{\beta-1}(\mu\xi)\sin(\mu\xi)\right] \\ &\quad + 3k^2\alpha^3\beta\mu\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi) = 0 \quad (23) \end{aligned}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$$\beta - 3 = 3\beta - 1 \quad , \quad \text{then} \quad \beta = -1$$

$$-\lambda^2\alpha\beta\mu - \alpha\beta^3\mu^3 k^4 = 0$$

$$k^4\alpha\beta(\beta-1)(\beta-2)\mu^3 + 3k^2\alpha^3\beta\mu = 0 \qquad (24)$$

By solving the algebraic system (24), we get,

$$\lambda = \mp i k^2 \mu \quad , \quad \alpha = \mp i\sqrt{2}k\mu \qquad (25)$$

Then by substituting (25) into Eq. (6) then, the exact soliton solution of equation (18) can be written in the form:

$$u(x,t) = \mp i\sqrt{2}k\mu \sec\left(\mu k(x \pm ik\mu t)\right)$$

Or

$$u(x,t) = \mp\sqrt{2}k\mu\,\mathrm{sech}\left(\mu k(ix \mp k\mu t)\right) \qquad (26)$$

For $k = \mu = 1$, Eq.(26) becomes

$$u(x,t) = \mp\sqrt{2}\,\mathrm{sech}(ix \mp t)$$

Section 4.3: Traveling Wave Solutions for a Generalized Kawahara and Hunter-Saxton Equations

Many partial differential equations which are of interest to study and investigate due to the role they play in various areas of mathematics and physics are included in this category. The formulation of classical theory of surfaces in a form familiar to the soliton theory, which makes possible an application of the analytical methods of this theory to integrable cases. When studying these nonlinear phenomena, which particularly plays a major role in many fields of physics, one can encounter with an equation of the form

$$u_t + ru^n u_x - qu_{5x} = 0. \qquad (27)$$

where r, p and q are nonzero arbitrary constants and $n = 1,2,3,\ldots$. When $n = 1$ and $n = 2$, Eq 27 is called Kawahara and modified Kawahara equations, respectively. In Eq. 27 the second term is convective part and the third term is dispersive part. The Kawahara and modified Kawahara equations have been the subject of extensive research work in recent decades.

Non-linear problems are more difficult to solve than linear ones. When studying these nonlinear phenomena, which particularly plays a major role in many fields of physics, such as fluid mechanics, solid state physics and plasma physics. Several direct methods have been recently proposed for obtaining exact and approximate analytic solutions for nonlinear evolution equations, such as homogeneous balance method 14 the modified cosine-function method 15 the extended cosine -function method 16, the cosine-function method 17 the Jacobi elliptic function expansion method 18and the sine-cosine method 19 and so on. The modified Kawahara equation also has wide applications in physics such as plasma waves, capillary-gravity water waves, water waves with surface tension, shallow water waves and so on 20-22

The Hunter - Saxton equation describes the propagation of waves in a massive director field of a nematic liquid cristal $(x,)$ being related to the deviation of the average orientation of the molecules from an equilibrium position 23 The Hunter - Saxton equation first appeared in 24 as an asymptotic equation for rotators in liquid crystals.

$$2u_x u_{2x} + u_{xxt} + uu_{xt} = 0, \quad (28)$$

where $u(x,t)$ is a real – valued function, are integrable models for the propagation of non-linear waves in $1 + 1$ dimension.

Rest of the paper is organized as follows. In section 2 sine-cosine method is produced. Section 3 is devoted to employed the method on the generalized Kawahara equation. In section 4 the new solution of The Hunter - Saxton equation by direct integration. Finally, some references are listed in the end.

Section 4.4: The sine - cosine method

Now we describe the sine - cosine method 19 for a given nonlinear evolution equation, say, in two variables 25-30,

$$f(u, u_x, u_t, u_{xx}, u_{xt}, \cdots) = 0, \quad (29)$$

Where $u(x,t)$ is traveling wave solution. We first consider it traveling wave solutions

$$u(x,t) = u(\xi), \quad \xi = (x - ct), \text{where } u_t = -cu_\xi, \ u_x = u_\xi, \ u_{2x} = u_{2\xi}, \ldots \quad (30)$$

Where c c is constant parameter to be determined, and k is an arbitrary constant, then equation (29) becomes an ordinary differential equation, which is integrated as long as all terms contain derivatives. The associated integration constants can be taken as zero. The next crucial step is that the solution we are looking for is expressed in the form

$$u(\xi) = \lambda \sin^{\beta}(\mu\xi), \quad and \quad \mu\xi \neq m\pi \quad m = 0, \pm 1, \pm 2, \ldots \quad (31)$$

Or

$$u(\xi) = \lambda \cos^{\beta}(\mu\xi), \quad and \quad \mu\xi \neq \frac{(2m+1)\pi}{2} \quad m = 0, \pm 1, \pm 2, \ldots \quad (32)$$

Where λ and μ are constants to be determined, μ and c are the wave number and the wave speed, respectively [20-23]. We use

$$u(\xi) = \lambda \sin^{\beta}(\mu\xi), \qquad u^{n+1} = \lambda^{n+1} \sin^{(n+1)\beta}(\mu\xi),$$

$$u_\xi = \lambda\beta \cos(\mu\xi) \sin^{\beta-1}(\mu\xi)$$

$$u_{2\xi} = -\lambda\beta M \sin^{\beta-1}(M\xi)\sin \ (M\xi) + \lambda M\beta(\beta-1)\sin^{\beta-2}(M\xi)\cos^2(M\xi)$$

$$= -\lambda M^2 \beta \sin^{\beta}(M\xi) + \lambda M^2\beta(\beta-1)\sin^{\beta-2}(M\xi)[1 - \sin^2(M\xi)]$$

$$= -\lambda M^2 \beta \sin \ (M\xi) + \lambda M^2\beta(\beta-1)\sin^{\beta-2}(M\xi) - \lambda M\beta(\beta-1)\sin^{\beta}(M\xi)$$

$$= -\lambda M^2\beta\sin^{\beta}(M\xi) + \lambda M^2\beta(\beta-1)\sin^{\beta-2}(M\xi) - \lambda M\beta^2\sin^{\beta}(M\xi) + \lambda M^2\beta\sin^{\beta}(M\xi)$$

$$u_{2\xi} = \lambda M^2\beta(\beta-1)\sin^{\beta-2}(M\xi) - \lambda M^2\beta^2\sin^{\beta}(M\xi)$$

$$u_{3\xi} = \lambda M^3\beta(\beta-1)(\beta-2)\sin^{\beta-3}(M\xi)\cos\ (M\xi) - \lambda M^3\beta^3\sin^{\beta-1}(M\xi)\cos\ (M\xi)$$

$$u_{4\xi} = -\lambda M^4\beta(\beta-1)(\beta-2)\sin^{\beta-3}(M\xi)\sin\ (M\xi) + \lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(M\xi)\cos^2(M\xi) + \lambda M^4\beta^3\sin^{\beta-1}(M\xi)\sin\ (M\xi) - \lambda M^4\beta^3(\beta-1)\sin^{\beta-2}(M\xi)\cos^2(M\xi)$$

$$= -\lambda M^4\beta(\beta-1)(\beta-2)\sin^{\beta-2}(M\xi) + \lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(M\xi)[1-\sin^2(M\xi)]$$

$$= -\lambda M^4\beta(\beta-1)(\beta-2)\sin^{\beta-2}(M\xi) + \lambda M^2\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(M\xi)$$

$$-\lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-2}(M\xi) + \lambda M^4\beta^3\sin^{\beta}(M\xi) - \lambda M^4\beta^3(\beta-1)\sin^{\beta-2}(M\xi) + \lambda M^4\beta^3(\beta-1)\sin^{\beta}(M\xi)$$

$$u_{2\xi} = \mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) - \mu^2\sin^{\beta}(\mu\xi),$$

$$u_{3\xi} = \mu^3\lambda\beta(\beta-1)(\beta-2)\cos(\mu\xi)\sin^{\beta-2}(\mu\xi) - \mu^3\lambda\beta^3\cos(\mu\xi)\sin^{\beta-1}(\mu\xi), (33)$$

$$u_{4\xi} = \mu^4\lambda\beta^4\sin^{\beta}(\mu\xi) - 2\mu^4\lambda\beta(\beta-1)(\beta^2-2\beta+2)\sin^{\beta-2}(\mu\xi) + \mu^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(\mu\xi)$$

$$u_{5\xi} = \mu^5\lambda^5\cos(\mu\xi)\sin^{\beta-1}(\mu\xi) - 2\mu^5\lambda\beta(\beta-1)(\beta-2)(\beta^2-2\beta+2)\cos(\mu\xi)\sin^{\beta-3}(\mu\xi) + \mu^5\lambda\beta(\beta-1)(\beta-2)(\beta-3)(\beta-4)\cos(\mu\xi)\sin^{\beta-5}(\mu\xi).$$

$$= -\lambda M^4\beta\sin^{\beta-2}(M\xi)[(\beta-2)+(\beta-2)(\beta-3)+\beta^2] - \lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(M\xi) + \lambda M^4\beta^3\sin^{\beta}(M\xi) + \lambda M^4\beta^4\sin^{\beta}(M\xi) - \lambda M^4\beta^3\sin^{\beta}(M\xi)$$

$$= \lambda M^4\beta\sin^{\beta-2}(M\xi)[\beta-2+\beta^2-3\beta+6+\beta^2]$$

$$= \lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(M\xi) + \lambda M^4\beta^4\sin^{\beta}(M\xi) - 2\lambda M^4\beta(\beta-1)[\beta^2-2\beta+2]\sin^{\beta-2}(M\xi)$$

$$u(\xi) = \lambda\cos^{\beta}(M\xi)$$

$$u_{\xi} = -\lambda\beta M\cos^{\beta-1}(M\xi)\sin\ (M\xi)$$

$$u_{2\xi} = -\lambda M^2\beta\cos^{\beta-1}(M\xi)\cos\ (M\xi) + \lambda M^2\beta(\beta-1)\cos^{\beta-2}(M\xi)\sin^2(M\xi)$$

$$= -\lambda M^2\beta\cos^{\beta}(M\xi) + \lambda M^2\beta(\beta-1)\cos^{\beta-2}(M\xi)[1-\cos^2(M\xi)]$$

$$= -\lambda M^2\beta\cos^{\beta}(M\xi) + \lambda M^2\beta(\beta-1)\cos^{\beta-2}(M\xi) - \lambda M^2\beta(\beta-1)\cos^{\beta}(M\xi)$$

$$= -\lambda M^2\beta\cos^{\beta}(M\xi) + \lambda M^2\beta(\beta-1)\cos^{\beta-2}(M\xi) - \lambda M^2\beta^2\cos^{\beta}(M\xi) + \lambda M^2\beta\cos^{\beta}(M\xi)$$

$$= \lambda M^2\beta(\beta-1)\cos^{\beta-2}(M\xi) - \lambda M^2\beta^2\cos^{\beta}(M\xi)$$

$$u_{3\xi} = -\lambda M^3\beta(\beta-1)(\beta-2)\cos^{\beta-3}(M\xi)\sin\ (M\xi) + \lambda M^3\beta^3\cos^{\beta-1}(M\xi)\sin\ (M\xi)$$

$$\begin{aligned} u_{4\xi} = &+\lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi)\sin^2(M\xi) \\ &- \lambda M^4\beta(\beta-1)(\beta-2)\cos^{\beta-3}(M\xi)\cos\ (M\xi) \\ &- \lambda M^4\beta^3(\beta-1)\cos^{\beta-2}(M\xi)\sin^2(M\xi) + \lambda M^4\beta^3\cos^{\beta-1}(M\xi)\cos\ (M\xi) \end{aligned}$$

$$\begin{aligned} = &\lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi)[1\cos^2(M\xi)] - \lambda M^4\beta(\beta-1)(\beta-2)\cos^{\beta-2}(M\xi) \\ &- \lambda M^4\beta^3(\beta-1)\cos^{\beta-2}(M\xi)[1\cos^2(M\xi)] + \lambda M^4\beta^3\cos^{\beta}(M\xi) \end{aligned}$$

$$\begin{aligned} = &\lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi) - \lambda M^4\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-2}(M\xi) \\ &- \lambda M^4\beta(\beta-1)(\beta-2)\cos^{\beta-2}(M\xi) - \lambda M^4\beta^3(\beta-1)\cos^{\beta-2}(M\xi) \\ &+ \lambda M^4\beta^3(\beta-1)\cos^{\beta}(M\xi) + \lambda M^4\beta^3\cos^{\beta}(M\xi) \end{aligned}$$

$$\begin{aligned} = &\lambda M\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi) \\ &- \lambda M^4\beta(\beta-1)[(\beta-2)(\beta-3) + (\beta-2) + \beta^2]\cos^{\beta-2}(M\xi) + \lambda M^4\beta^4\cos^{\beta}(M\xi) \\ &- \lambda M^4\beta^3\cos^{\beta}(M\xi) + \lambda M^4\beta^3\cos^{\beta}(M\xi) \end{aligned}$$

$$\begin{aligned} = &\lambda M\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi) - 2\lambda M^4\beta(\beta-1)[\beta^2-2\beta+2]\cos^{\beta-2}(M\xi) \\ &+ \lambda M^4\beta^4\cos^{\beta}(M\xi) \end{aligned}$$

The parameter β will be found by balancing the highest-order nonlinear terms with the highest-order partial derivative term in the given equation and then give the formal solution. Substituting the formal solution 31 or 32

into the ordinary differential equation obtained above and the change it into hyperbolic polynomial identities for the intermediate variable ξ. Collect all terms with the same power in $\sin^k(\xi)$ and $\cos(\mu\xi)\sin^k(\mu\xi)$ or $\cos^j(\mu\xi)$ and $\sin(\mu\xi)\cos^j(\mu\xi)$ and set to zero their coefficients to get algebraic relations among the unknowns c, λ, μ, β. With the aid of Mathematica and using the Wu's elimination method [31], we solve for the above unknowns of the equations to finally obtain the traveling wave solutions of the given nonlinear evolution equation.

4. Traveling wave solutions for a generalized Kawahara equation

For the generalized Kawahara equation (1) which contains some particular important equations such as Kawahara and modified Kawahara equations. In order to get a new traveling wave solutions of equation (1). Substituting (4) into (1), we obtain an ordinary differential equation

$$-cu_{\xi} + ru^n u_{\xi} + pu_{3\xi} - qu_{5\xi} = 0 \qquad (34)$$

Integrating Eq. (34) and integration constant can be taken as zero, we find

$$-cu + \frac{r}{n+1}u^{n+1} + pu_{2\xi} - qu_{4\xi} = 0 \qquad (35)$$

Substituting Eq. (7) into Eq. (35), we obtain

$$\begin{aligned} -c\lambda\sin^{\beta}(\mu\xi) + \frac{r}{n+1}\lambda^{n+1}\sin^{(n+1)\beta}(\mu\xi) + p\mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) - p\mu^2\lambda\beta^2\sin^{\beta}(\mu\xi) \\ - q\mu^4\beta^4\sin(\mu\xi) + 2q\mu^4\lambda\beta(\beta-1)(\beta^2-2\beta+2)\sin^{\beta-2}(\mu\xi) \\ - q\mu^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)\sin^{\beta-4}(\mu\xi) = 0 \qquad (36) \end{aligned}$$

Setting the coefficients of $\sin^{\beta}\mu\xi, \sin^{\beta-2}\mu\xi$ to zero and equating the exponents of $\sin^{(n+1)\beta}\mu\xi$, and $\sin^{\beta-4}\mu\xi$, we find the following equations:

$$-c\lambda - p\mu^2\lambda\beta^2 - q\mu^4\lambda\beta^4 = 0,$$

$$p\mu^2\lambda\beta(\beta-1) + 2q\mu^4\lambda\beta(\beta-1)(\beta^2-2\beta+2) = 0$$

$$\beta - 4 = n\beta + \beta \qquad (37)$$

$$\frac{r}{n+1}\lambda^{n+1} - q\mu^{44}\lambda\,\beta(\beta-1)(\beta-2)(\beta-3) = 0$$

We now solve the above set of equations (2) by using the Wu's elimination method, and obtain the following solutions:

$$\beta = -\frac{4}{n},$$

$$-c\lambda - \rho M^2\lambda\beta^2 - qM^4\lambda\beta^4 = 0$$

$$c = -\rho M^2\beta^2 - qM^4\beta^4$$

$$= -\rho\left(\frac{-\rho}{2q(\beta^2-2\beta+2)}\right)\left(\frac{-4}{n}\right)^2 - q\left(\frac{\rho^2}{q(\beta^2-2\beta+2)}\right).\left(\frac{-4}{n}\right)^2$$

$$= -\rho\left[\frac{n^2}{2^2}\frac{\rho}{q(8+4n+n^2)}.\frac{16}{n^2} - q\frac{n^4}{16}.\frac{\rho^2}{q^2(8+4n+n^2)^2}\right]$$

$$= \frac{4\rho^2}{q(8+4n+n^2)} - \frac{16\rho^2}{q(8+4n+n^2)^2}$$

$$= \frac{4\rho^2(8+4n+n^2) - 16\rho^2}{q(8+4n+n^2)^2}$$

$$= \frac{4\rho^2(8+4n+n^2-4)}{q(8+4n+n^2)^2} = \frac{4\rho^2(8+4n+n^2)}{q(8+4n+n^2)^2}$$

$$c = \frac{4p^2(2+n)^2}{q(8+4n+n^2)^2},$$

$$-c\lambda - \rho M^2\lambda\beta^2 - qM^4\lambda\beta^4 = 0$$

$$\rho M^2\lambda\beta(\beta-1) + 2qM^4\lambda\beta(\beta-1)(\beta^2-2\beta+2) = 0$$

$$= qM^4(\beta^2-2\beta+2) = -\rho M^2$$

$$M^2 = \frac{-\rho}{2q(\beta^2-2\beta+2)} = \frac{-\rho}{q\left(\frac{16+8n+2n}{n^2}\right)} = \frac{-\rho}{q\left(\frac{16+8n+2n}{n^2}\right)}$$

$$M=\sqrt{\frac{-\rho}{2q(\beta^2-2\beta+2)}}=\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$\mu=\frac{n}{2}\sqrt{\frac{-p}{q(8+4n+n^2)}}, \qquad (38)$$

$$\frac{\eta}{n+1}\lambda^{n+1}-qM^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)=0$$

$$\frac{\eta}{n+1}\lambda^{n+1}=qM^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)$$

$$\frac{\eta}{n+1}\lambda^{n}=qM^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)$$

$$=q\frac{n^4}{16}\frac{\rho^2}{q^2(8+4n+n^2)^2}-\frac{4}{n}\left[\frac{-4}{n}-1\right]\left[\frac{-4}{n}-2\right]\left[\frac{-4}{n}-3\right]$$

$$=\frac{n^4}{16}\frac{\rho^2}{q(8+4n+n^2)^2}+\frac{4}{n}\left[\frac{4+n}{n}\right]\left[\frac{4+2n}{n}\right]\left[\frac{4+3n}{n}\right]$$

$$=\frac{n^4}{16}\frac{\rho^2}{q(8+4n+n^2)^2}\frac{4}{n^4}[4+n][4+2n][4+3n]$$

$$\lambda^n=\frac{4\rho^2[n+1][4+2n][4+3n][4+n]}{16rq(8+4n+n^2)^2}=\frac{2c[n+1][n+4][4+3n]}{16r[n+2]}$$

$$\lambda^n=\frac{c[n+1][n+4][4+3n]}{8r[n+2]}$$

$$\lambda=\left\{\frac{c(n+1)(4+n)(4+3n)}{8r(n+2)}\right\}^{\frac{1}{n}}.$$

$$-cu\frac{\eta}{n+1}u^{n+1}+pu_{2\xi}-qu_{5\xi}=0$$

$$u(\xi)=\lambda\cos(M\xi),u_{2\xi}=M^2\lambda\beta(\beta-1)\cos^{\beta-2}(M\xi)-M^2\lambda\beta^2\cos^{\beta}(M\xi)$$

$$u_{4\xi}=-M^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi)-M^4\lambda\beta(\beta-1)\cos^{\beta-2}(M\xi)[\beta^2-\beta+2]$$

Substituting $u,u_{2\xi},u_{4\xi}$ into equation 8i

$$-c\left(\lambda\cos^{\beta}(M\xi)\right)+\frac{\eta}{n+1}\lambda^{n+1}\cos^{(n+1)\beta}(M\xi)+p\left[M^2\lambda\beta(\beta-1)\cos^{\beta-2}(M\xi)-M^2\lambda\beta^2\cos^{\beta}(M\xi)\right]$$
$$-q\left[-M^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(M\xi)-M^4\lambda\beta(\beta-1)\right][\beta^2-2\beta$$
$$+2]\cos^{\beta-2}(M\xi)+M^4\lambda\beta^4\cos^{\beta}(M\xi)$$

Setting the coefficients of$\cos^{\beta}(M\xi)$,$\cos^{\beta-2}(M\xi)$ to zero and equating the exponents $\cos^{\beta(n+1)}(M\xi)$ and $\cos^{\beta-4}(M\xi)$we find the following equqqations

$$\cos^{\beta}(M\xi) = -c\lambda - M^2\lambda\beta^2 - M^4\lambda\beta^4 = 0$$

$$\cos^{\beta-2}(M\xi) = pM^2\lambda\beta(\beta-1) + qM^4\lambda\beta(\beta-1)(\beta^2-2\beta+2) = 0$$

$$n\beta + \beta = \beta - 4$$

$$\frac{\eta}{n+1}\lambda^{n+1} - qM^4\lambda B(\beta-1)[\beta^2-2\beta+2] = 0$$

We now solve the above set of equation (5i) by using the wu,s elimination method, and obtain the following

$$n\beta + \beta = \beta - 4$$

$$\beta = \frac{-4}{n}$$

$$\rho M^2\lambda\beta(\beta-1) + 2qM^4\lambda\beta(\beta-1)(\beta^2-2\beta+2) = 0$$

$$qM^2 = -\frac{\rho}{(\beta^2-2\beta+2)}, \qquad M^2 = \frac{-\rho}{2q(\beta^2-2\beta+2)}$$

$$= \frac{-\rho}{2q\left(\frac{16}{n^2}+\frac{8}{n}+2\right)} = \frac{-\rho}{2q\left(\frac{16+8n+2n^2}{n^2}\right)}$$

$$\frac{-\rho}{4q\left(\frac{8+4n+n^2}{n^2}\right)} = \frac{n^2}{4q}\frac{-\rho}{(8+4n+n^2)}$$

$$M = \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}} = \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$-c\lambda - \rho M^2\lambda\beta - qM^4\lambda\beta^4 = 0$$

$$c\lambda = -\rho M^2\lambda\beta^2 - qM^4\lambda\beta^4$$

$$c = \frac{n^2}{4q}\frac{-\rho}{(8+4n+n^2)}\frac{(-4)^2}{n^2} - q\frac{n^4}{16}\frac{1}{q}\frac{\rho^2}{(8+4n+n^2)}\frac{(4)^4}{n^2}$$

$$\frac{\rho^2 4}{q(8+4n+n^2)} - \frac{16\rho^2}{q(8+4n+n^2)}$$

$$\frac{4\rho^2(8+4n+n^2) - 16\rho^2}{q(8+4n+n^2)^2} - \frac{4\rho^2(8+4n+n^2-4)}{q(8+4n+n^2)^2}$$

$$\frac{4\rho^2(8+4n+n^2+4)}{q(8+4n+n^2)^2} = \frac{4\rho^2[2+n]^n}{q(8+4n+n^2)}$$

$$\frac{\eta}{n+1}\lambda^{n+1} - qM^4\lambda\beta(\beta-1)(\beta-2)(\beta-3) = 0$$

$$\frac{\eta}{n+1}\lambda^{n} = qM^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)$$

$$\frac{\rho^2}{q(8+4n+n^2)} = q\frac{n^4}{16} - \frac{4}{n}\left[\frac{-4}{n}-1\right]\left[\frac{-4}{n}-2\right]\left[\frac{-4}{n}-3\right]$$

$$= \frac{\rho^2 q}{2q(8+4n+n^2)}\frac{n^4}{16}\frac{4}{n}\left[\frac{4+n}{n}\right]\left[\frac{4+2n}{n}\right]\left[\frac{4+3n}{n}\right]$$

$$= -\frac{qn^4}{16}\frac{\rho^2}{q^2(8+4n+n^2)^2}\frac{4}{n^4}[4+n][4+2n][4+3n]$$

$$= -\frac{4\rho^2[4+n][4+2n][4+3n]}{16q(8+4n+n^2)^2}$$

$$\lambda^n = -2c\frac{[n+1][4+n][4+3n]}{16q[2+n]}$$

$$-\frac{c[n+1][4+n][4+3n]}{8r[2+n]}$$

$$\lambda = \frac{c[n+1][4+n][4+3n]^{\frac{1}{n}}}{8r[2+n]}$$

We find the following four types of traveling wave solutions for the generalized Kawahara equation 1

Type 1. For $pq > 0$

$$u(x,t) = \lambda\sin^{\beta}(M\xi)$$

$$\lambda = \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}}$$

$$M = \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$\xi = x - ct + k$$

$$\beta = \frac{-4}{n}$$

Type I for $pq > 0$

$$(-i)^{\frac{4}{n}}\left[\frac{c[n+1][n+4][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}}\sinh^{\frac{-4}{n}}\left[\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}\right]$$

$$= (-i)^{\frac{4}{n}} \left[\frac{c[n+1][n+4][4+3n]}{8r[n+2]}\right]^{\frac{1}{n}} \frac{1}{\sinh^{\frac{4}{n}}} \left[\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)\right]$$

$$u_1(x,t) = (-i)^{\frac{4}{n}} \left\{\frac{c(n+1)(4+n)(4+3n)}{8r(n+2)}\right\}^{\frac{1}{n}} \operatorname{csch}^{\frac{4}{n}} \left[\frac{n}{2}\sqrt{\frac{-p}{q(8+4n+n^2)}}(x-ct+k)\right] \quad (39)$$

Type 2. For $pq < 0$

$$u_2(x,t) = \lambda \cosh^{\beta}(M\xi)$$

$$\lambda = \left[\frac{c[n+1][4+n][4+3n]}{8r[n+2]}\right]^{\frac{1}{n}} \quad M = \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$\beta = \frac{-4}{n}, \xi = x - ct + k$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \frac{1}{\cosh^{\frac{-4}{n}}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}} x - ct + k$$

$$\left[\frac{c[n+1][4+n][4+3n]}{8r}\right]^{\frac{1}{n}} \operatorname{sech}^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}} x - ct + k$$

$$u_2(x,t) = \left\{\frac{c(n+1)(4+n)(4+3n)}{8r(n+2)}\right\}^{\frac{1}{n}} \operatorname{sech}^{\frac{4}{n}} \left[\frac{n}{2}\sqrt{\frac{-p}{q(8+4n+n^2)}}(x-ct+k)\right] \quad (40)$$

Type 3. For $pq < 0$

$$u_2(x,t) = \lambda \cos^{\beta}(M\xi)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \cos^{\frac{-4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \frac{1}{\cos^{\frac{4}{n}}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \sec^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$\left(\frac{x - 4\rho^2[2+n]^n t + k}{q(8+4n+n^2)^2}\right)$$

$$\left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \sec^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}$$

$$\left(\frac{x-4\rho^2[2+n]^n t+k}{q(8+4n+n^2)^2}\right)$$

$$u_3(x,t) = \left\{\frac{c(n+1)(4+n)(4+3n)}{8r(n+2)}\right\}^{\frac{1}{n}} \sec^{\frac{4}{n}}\left[\frac{n}{2}\sqrt{\frac{-p}{q(8+4n+n^2)}}(x-ct+k)\right] \quad (41)$$

Type 4. For $pq < 0$

$$u(x,t) = \lambda \sin^{\beta}(M\xi)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \sin^{\frac{-4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \frac{1}{\sin^{\frac{4}{n}}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+x-ct+k)$$

$$= \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \csc^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}\left(\frac{x-4\rho^2[2+n]^n t+k}{q(8+4n+n^2)^2}\right)$$

$$u_4(x,t) = \left\{\frac{c(n+1)(4+n)(4+3n)}{8r(n+2)}\right\}^{\frac{1}{n}} \csc^{\frac{4}{n}}\left[\frac{n}{2}\sqrt{\frac{-p}{q(8+4n+n^2)}}(x-ct+k)\right] \quad (42)$$

From the above solution of generalized Kawahara equation, I can find the solutions of Kawahara and modified Kawahara equations as follows:

(1) For $n = 1$, we obtain the Kawahara equation in the form

$$u_t + ruu_x + pu_{3x} - qu_{5x} = 0, \quad (43)$$

and the solitary wave solutions of Kawahara equation are:

Type 1. For $pq > 0$

$$u_5(x,t) = (-i)^{\frac{4}{n}}\left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \csc^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

For n=1=

$$= (-i)^4 \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \operatorname{csch}^{\frac{4}{n}n} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4+1)}}\left(\frac{x-4\rho^2[2+1]^2 t+k}{q(8+4+1)^2}\right)$$

$$= 1\,\frac{[c[2][0][7]]}{8\times 3} \operatorname{csch}^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(13)}}(x - \frac{36\rho^2}{169q} t + k$$

$$u_5(x,t) = \frac{35c}{12r} \operatorname{csch}^4 \left[\frac{1}{2}\sqrt{\frac{-p}{13q}}\left(x - \frac{36p^2}{169q} t + k\right)\right]. \qquad (44)$$

Type 2. For $pq > 0$

$$u_2(x,t) = \left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \operatorname{sech}^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$= c\left[\frac{(1+1)(4+1)(4+3\times 1)}{8r(1+2)}\right] \operatorname{sech}^{\frac{4}{1}} \frac{1}{2}\sqrt{\frac{-\rho}{q(8+4+1)}}x - \frac{4\rho^2[2+1]^1}{(8+4+1)^2} t + k$$

$$u_6(x,t) = \frac{35c}{12r} \operatorname{sech}^4 \left[\frac{1}{2}\sqrt{\frac{-p}{13q}}\left(x - \frac{36p^2}{169q} t + k\right)\right]. \qquad (45)$$

Type 3. For $pq < 0$

$$u_7(x,t) = \left[\frac{c[n+1][4+n][4+3n]}{8\eta[2+n]}\right]^{\frac{1}{n}} \csc^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$\left[\frac{c[1+1][4+1][4+3]}{8r[2+1]}\right]^{\frac{1}{1}} \csc^{\frac{4}{1}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$= \frac{35c}{12r} \csc^4 \frac{1}{2}\sqrt{\frac{-\rho}{13q}}x - \frac{4\rho^2[2+n]^n}{q(8+4n+n^2)^2} t + k \qquad (46)$$

$$u_7(x,t) = \frac{35c}{12r} \csc^4 \left[\frac{1}{2}\sqrt{\frac{-p}{13q}}\left(x - \frac{36p^2}{169q} t + k\right)\right]. \qquad (47)$$

Type 4. For $pq < 0$

$$u_8(x,t) =$$

$$\left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \sec^{\frac{4}{n}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$$\left[c\frac{[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \sec^{\frac{4}{1}} \frac{n}{2}\sqrt{\frac{-\rho}{q(8+4+1)}}\left(\frac{x-4\rho^2[2+1]^2 t}{q(8+4+1)^2}+k\right)$$

$$=\frac{35c}{12r}\csc^4\sec^{\frac{4}{n}}\frac{n}{2}\sqrt{\frac{-\rho}{13q}}x-\frac{4\rho^2 xq}{169q}t+k)$$

$$u_8(x,t)=\frac{35c}{12r}\sec^4\left[\frac{1}{2}\sqrt{\frac{-p}{13q}}\left(x-\frac{36p^2}{169q}t+k\right)\right]. \qquad (48)$$

(2) For $n=2$, we obtain the modified Kawahara equation in the form

$$u_t+ru^2u_x+pu_{3x}-qu_{5x}=0, \qquad (49)$$

we obtain the following four types of solitary wave solutions

Type 1. For $pq>0$

$$u(x,t)=(-i)^{\frac{4}{n}}\left[\frac{c[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}}\csc^{\frac{4}{n}}\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

For n=2

$$(-i)^{\frac{4}{2}}\left[c\frac{[2+1][4+2][4+3\times 2]}{8r[2+2]}\right]^{\frac{1}{2}}\operatorname{csch}^{\frac{4}{2}}\frac{2}{2}\sqrt{\frac{-\rho}{q(8+4\times 2+2^2)}}x-ct+k$$

$$u_9(x,t)=\left[\frac{c[3][6][10]}{8r[4]}\right]^{\frac{1}{2}}\operatorname{csch}^2\sqrt{\frac{-\rho}{20q}}\left(\frac{x-4\rho^2[2+2]^2 t}{q(8+4\times 2+2^2)^2}+k\right)$$

$$=\sqrt{\frac{45c}{8r}}\operatorname{csch}^2\sqrt{\frac{-\rho}{20q}}\left(x-\frac{4\rho^2(16)}{q(400)}t+k\right)$$

$$u_9(x,t)=-\sqrt{\frac{45c}{8r}}\operatorname{csch}^2\left[\sqrt{\frac{-p}{20q}}\left(x-\frac{4p^2}{25q}t+k\right)\right]. \qquad (50)$$

Type 2. For $pq>0$

$$u_{10}(x,t) = \left[c\frac{[n+1][4+n][4+3n]}{8r[2+n]}\right]^{\frac{1}{n}} \operatorname{sech}^{\frac{4}{n}}\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}(x-ct+k)$$

$n = 2$

$$= \left[\frac{c[2+1][4+2][4+3\times 2]}{8r[2+2]}\right]^{\frac{1}{2}} \operatorname{sech}^{\frac{4}{2}}\frac{2}{2}\sqrt{\frac{-\rho}{q(8+4\times 2+2^2)}}\left(\frac{x-4\rho^2[2+2]^2}{q(8+8+4)^2}t+k\right)$$

$$= \left[\frac{c[3][6][10]}{8r[4]}\right]^{\frac{1}{2}} \operatorname{sech}^2\sqrt{\frac{-\rho}{20q}}\left(\frac{x-4\rho^2[16]}{400q}t+k\right)$$

$$= \sqrt{\frac{45c}{8r}}\operatorname{sech}^2\sqrt{\frac{-\rho}{20q}}\left(x-\frac{4\rho^2}{q}t+k\right)$$

$$u_{10}(x,t) = -\sqrt{\frac{45c}{8r}}\operatorname{sech}^2\left[\sqrt{\frac{-p}{20q}}\left(x-\frac{4p^2}{25q}t+k\right)\right]. \qquad (51)$$

Type 3. For $pq < 0$

(7)$u_{11}(x,t) =$

$$\left[\frac{c[n+1][4+n][4+3n]}{8r[n+2]}\right]^{\frac{1}{n}} \csc^{\frac{4}{n}}\frac{n}{2}\sqrt{\frac{-\rho}{q(8+4n+n^2)}}x-ct+k$$

$$\left[\frac{c[2+2][4+2][4+3\times 2]}{8r[2+2]}\right]^{\frac{1}{2}} \csc^{\frac{4}{2n}}\frac{2}{2}\sqrt{\frac{-\rho}{q(8+8+4)}}\left(\frac{x-4\rho^2[2+2]^2}{q(8+8+4)^2}t+k\right)$$

$$= \left[\frac{c[3][6][10]}{4.8r}\right]^{\frac{1}{2}} \csc^2\sqrt{\frac{-\rho}{20q}}\left(\frac{x-4\rho^2[4]^2}{20^2q}t+k\right)$$

$$u_{11}(x,t) = -\sqrt{\frac{45c}{8r}}\csc^2\left[\sqrt{\frac{-p}{20q}}\left(x-\frac{4p^2}{25q}t+k\right)\right]. \qquad (52)$$

For 4. For $pq < 0$

$$u_{12}(x,t) = -\sqrt{\frac{45c}{8r}}\sec^2\left[\sqrt{\frac{-p}{20q}}\left(x-\frac{4p^2}{25q}t+k\right)\right]. \qquad (53)$$

Section 4.6: Exact solution for Hunter - Saxton equation

I now consider the Hunter - Saxton equation (2), by using the traveling wave solutions $u(x,t) = u(\xi),\ \ \xi = (x - ct + k)$ in Eq. (2) and I obtained nonlinear ordinary differential equation in the form 32-38

$$2u'u'' - cu''' + uu''' = 0, \qquad (54)$$

by integrating Eq. (54) and integration constant can be taken as zero, we obtain

$$\frac{u'^2}{2} + (u - c)u'' = 0, \qquad (55)$$

multiply Eq. (55) by $\frac{1}{\sqrt{u-c}}$, I obtain

$$\frac{v'^2}{2\sqrt{u-c}} - v''(u-c)\frac{1}{\sqrt{u-c}} = 0$$

$$\frac{1}{2\sqrt{u-c}}v'^2 - \frac{v''(u-c)}{\sqrt{u-c}} \times \frac{\sqrt{u-c}}{\sqrt{u-c}} = 0$$

$$= \frac{1}{\sqrt{u-c}}v'^2 - \frac{v''(u-c)\sqrt{u-c}}{u-c} = 0$$

I can write Eq. (4)

$$\left(\sqrt{u-c}v'\right)' = 0$$

$$= \left((u-c)^{1/2}v'\right)' = 0$$

$$\frac{1}{2\sqrt{u-c}}u'^2 + \sqrt{u-c}u'' = 0, \qquad (56)$$

I can write Eq. 56in the form

$$\left[\sqrt{u-c}u'\right]' = 0, \qquad (57)$$

by integrating Eq. 57 I obtain the solution of Hunter - Saxton equation as the form

$$(u-c)^{1/2}\frac{\partial u}{\partial \xi} = 0$$

$$(u-c)^{1/2}\frac{\partial u}{\partial \xi} = \partial_1$$

$$= \frac{2}{3}(u-c)^{\frac{2}{3}} = \partial_1\xi + \partial_2$$

$$\frac{2}{3} \times \frac{3}{2}(u-c)^{\frac{3}{2}\times\frac{2}{3}} = \left(\frac{2}{3}(\partial_1\xi + \partial_2)\right)^{\frac{2}{3}}$$

$$= u - c = \left(\frac{2}{3}(\partial_1 \xi + \partial_2)\right)^{\frac{2}{3}}$$

$$u = \left(\frac{2}{3}(\partial_1 \xi + \partial_2)\right)^{\frac{2}{3}} + c$$

$$\xi = x - ct + k$$

$$u_{13} = (d_1(x - ct + k) + d_2)^{\frac{2}{3}} + c, \quad \text{where} d_1 \text{and} d_2 \text{are constants.} \quad (58)$$

5. Result

We obtain a new traveling wave solutions for the generalized Kawahara and Hunter - Saxton equations. With the aid of a symbolic computation system, three types of more general traveling wave solutions (including hyperbolic functions, trigonometric functions and radical functions) with free parameters are constructed. Solutions concerning solitary and periodic waves are also given by setting the three arbitrary parameters, involved in the traveling waves, as special values. The obtained results show that sine - cosine method is very powerful and convenient mathematical tool for nonlinear evolution equations in science and engineering.

6. Discussion

We use sine- cosine method to obtain a new traveling wave solutions of a generalized Kawahara equation and I obtain a new exact solution for Hunter - Saxton equation by direct integration. The traveling wave solutions for the above equation are presented to obtain novel exact solutions for the same equations. These solutions plays an important role in the modeling of many physical phenomena such as plasma waves, magento - acoustic wave and nematic liquid crystals.

7. Conclusion and Recommendation

Conclusion

The extended tanh-method is further extended into the nonlinear evolution equations whose balancing numbers may be any nonzero real numbers. On the other hand, by making use of a new general ans¨atze, we explore some new travelling wave solutions for certain physically interesting models described by nonlinear evolution equations. Particularly, for our analysis we have considered the 2+1dimensional generalization of Boussinesq equation13 and the system of variant Boussinesq equations14,15 and the generalized BBM17−25 equation. It should be noted that this method is a combination of extended tanh-method and its modification, and the results by this method not only cover the solutions obtained by the above method, but also some new formal solutions which cannot be obtained by extended -method and its modification. Moreover, because it is computerizable, the method allows us to perform the complicated and tedious algebraic calculations on a computer.

The sine-cosine function method has been implemented to find the solution for nonlinear partial differential equations. New exact solutions were found by the proposed method. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solutions and other areas.

The tanh, sech, and sine-cosine function methods have been successfully applied to find the solution for nonlinear partial differential equations. We can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solutions and other areas.

We obtain a new traveling wave solutions for the generalized Kawahara and Hunter - Saxton equations. With the aid of a symbolic computation system, three types of more general traveling wave solutions (including hyperbolic functions, trigonometric functions and radical functions) with free parameters are constructed. Solutions concerning solitary and periodic waves are also given by setting the three arbitrary parameters, involved in the traveling waves, as special values. The obtained results show that sine - cosine method is very powerful and convenient mathematical tool for nonlinear evolution equations in science and engineering.

Recommendation

We recommend to use the sine — cosine method to solve fractional differential equations and integral equations.

8. References

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